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RESEARCH ARTICLE

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# The modified mixture of distributions model: a revisit

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**Abstract** Andersen (J Financ **51**, 169–204 (1996)) introduced a modification of the mixture of distributions model based on microstructure arguments. Based on a small sample of five stocks, he infers that this modified mixture of distributions (MMD) model adequately captures the joint behavior of trading volume and volatility. We re-examine this claim using a larger sample of twenty-two stocks and two sample periods. Our tests show that 59% of the sample rejects the MMD model in the period 1973–1991, the same period studied by Andersen. Results for the second period (1993–1999) are more supportive of the MMD, especially for number of trades, although nearly one-third of the sample still rejects the MMD. We conclude that further tests are needed before the general validity of the MMD can be established.

**Keywords** Mixture of distributions · Volatility · Trading volume

**JEL Classification Numbers** C12 · C52

## 1 Introduction

The mixture of distributions model (MDM) has a long history in financial economics. Interest in the model continues to be seen in recent "structural" studies of the volume–volatility relationship (Jones, Kaul, and Lipson 1994; Easley, Kiefer, and

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O'Hara 1997) as well as research on the microeconomic foundations of ARCH models (Anderson 1996; Liesenfeld 2001; Fleming et al. 2001).

The central idea behind the MDM – that information arrivals drive trades and returns volatility – is intuitively appealing. However, evidence on its empirical validity has been elusive. While early tests by Epps and Epps (1976) and Tauchen and Pitts (1983) produced evidence that is consistent with the model, recent studies have rejected the model's implications (Richardson and Smith 1994). In an important paper, Andersen (1996) argues that these studies may have rejected the MDM due to misspecification of trading volume. Using microstructure arguments, he derived a version of the MDM in which volume follows a conditional Poisson process rather than a conditional normal process as assumed in prior research e.g., Richardson and Smith (1994). He then tested the moment implications of this modified mixture of distributions (MMD) model using the generalized method of moments (GMM). Based on a sample of five actively traded stocks for the period 1973–1991, the GMM over-identifying test rejects the model for only one stock at 5% and none at 1%. Andersen (1996) thus concludes that the model "provides an acceptable characterization of *the daily volume–volatility relationship*" (p. 171, italics added). We argue that this conclusion may be premature. Since Andersen's sample is very small, it is not clear how general his findings are. In particular, do they hold for most liquid stocks? And, is the model robust over time? We answer these questions by applying the same methodology as Andersen on a larger sample of 22 actively traded stocks and over two sample periods: 1973–1991 and 1993–1999. The first sample period is the same as the one used in Andersen (1996). The second sample period provides a check on the robustness of the MMD over time.

Our main results are as follows. First, we find that the MMD is rejected for most stocks in the first period. Therefore, we disagree with Andersen that the MMD provides an acceptable characterization of the volume-volatility relationship. Second, and on more encouraging note, the data for the second period is more consistent with the MMD. Nonetheless, one third of the sample still rejects the model, indicating that further tests are needed before we can establish the validity and robustness of the MMD model. Finally, our results are somewhat more supportive of the MMD when number of trades rather than total trading volume is used as the information arrival proxy. This is consistent with the findings of Jones et al. (1994) and Ane and Geman (2000) that number of trades is the relevant subordinator for stock prices.

The rest of this paper is organized as follows. Section 2 introduces the MMD. Section 3 discusses the moment restrictions of the MMD for returns and volume which are testable by GMM. Section 4 describes the dataset and volume de-trending issues. The main results of this paper are presented in Section 5. Section 6 examines whether our results are influenced by the volume de-trending method. Section 7 concludes the paper.

## 2 The modified mixture of distributions model

Andersen (1996) proposed a mixture of distributions model that strongly resembles the classic model of Clark (1973). The basic setting is as follows. Consider a market where stock prices move from one temporary equilibrium to another during the day in response to news arrivals. Let the total number of information arrivals on day $t$ be $J_t$, where $J_t$ is random. Let the maximum number of information arrivals on any single day be $J$. Let $K_t$ be a positive scaling factor representing the intensity of information arrivals. Define $J_t = K_t J$. For large $J$, Andersen shows that the stock returns ($R_t$) conditional on $K_t$ can be written as:

$$R_t | K_t \sim N(0, \sigma^2 K_t) \tag{1}$$

Trading volume is distributed as a binomial random variable – the probability that an informed agent trades on the new information is less than one. Therefore, the number of informed trades resulting from an information event will be less than the number of informed agents. Andersen approximates the binomial distribution with the Poisson distribution, and writes the conditional trading volume as:

$$V_t | K_t \sim c \cdot Po(m_0 + m_1 K_t) \tag{2}$$

where $c$ is an unknown positive scaling factor, $m_0$ is the noise component of volume and $m_1 K_t$ is the informed component of volume, and is assumed to be proportional to the information flow.

Two features distinguish this volume specification from the standard mixture of distribution model. First, in the MMD, volume is assumed to follow a conditional Poisson distribution rather than a conditional normal distribution. The Poisson specification respects the non-negativity of volume. Second, volume is decomposed into a noise component (with constant arrival intensity) and an informed trading component that varies with the information flow. This is in line with standard microstructure models with asymmetric information. Moreover, the probability of informed trading is conveniently captured by the parameter $1-cm_0$, which provides a simpler way to gauge the extent of informed trading compared to other measures advocated in the literature (Easley, Kiefer, O'Hara, and Paperman 1996; Easley et al. 1997).

## 3 Moment implications of the mmd model

Following Andersen, we test the MMD under minimal assumptions about the information arrival process. We do so by evaluating the joint unconditional moments of volume and returns as implied by the model using GMM. Let $X_t = (R_t, V_t)$ be the vector of observables comprising returns ($R_t$) and trading volume ($V_t$). Let $g_T(\theta)$ be the vector of sample moments to be tested based on a sample of $T$ observations:

$$g_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} h(X_t, \theta) \tag{3}$$

where the $L$-vector, $h(.)$, comprises functions of unconditional moments implied by the MMD, and $\theta$ is an $M$-vector of parameters governing the MMD. The idea behind GMM estimation is to find $\widehat{\theta}$ such that $g_T(\theta)$ is equal to zero. Following Andersen (1996), we use the following set of unconditional moment conditions to estimate the unknown parameters:[1]

[1] The set of moment equations used here is slightly different from that given in Andersen, (1996, p. 188). The derivation of the set of 12 moment conditions can be obtained upon request.

$$
\begin{aligned}
&E[R_t] = \bar{r} \\
&E[R_t - \bar{r}] = (2/\pi)^{1/2} E[K_t^{1/2}] \\
&E[(R_t - \bar{r})^2] = E[K_t] \equiv \bar{K} \\
&E|R_t - \bar{r}|^3 = 2(2/\pi)^{1/2} E[K_t^{3/2}] \\
&E[(R_t - \bar{r})^4] = 3E[\bar{K}^2 + \mathrm{var}(K_t)] \\
&E[\hat{V}_t] = c(m_0 + m_1\bar{K}) \equiv \bar{V} \\
&E[(\hat{V}_t - \bar{V})^2 = c\bar{V} + c^2 m_1^2 \mathrm{var}(K_t) \\
&E[(\hat{V}_t - \bar{V})^3 = c^2\bar{V} + 3c^3 m_1^2 \mathrm{var}(K_t) + c^3 m_1^3 E[K_t - \bar{K}]^3 \\
&E[R_t\hat{V}_t] = \bar{r}\bar{V} \\
&E[|R_t - \bar{r}|(\hat{V}_t - \bar{V})] = c(2/\pi)^{1/2} m_1 (E[K_t^{3/2}] - \bar{K}E[K_t^{1/2}]) \\
&E[(R_t - \bar{r})^2\hat{V}_t] = \bar{V}\bar{K} + cm_1\mathrm{var}(K_t) \\
&E[(R_t - \bar{r})^2(\hat{V}_t - \bar{V})^2] \\
&\quad = c\bar{K}\bar{V} + c^2 m_1 \mathrm{var}(K_t) + c^2 m_1^2[E[K_t - \bar{K}]^3 + \bar{K}\mathrm{var}(K_t)]
\end{aligned}
$$

Since we have 9 parameters and 12 moment conditions, this system is over-identified. However, Hansen (1982) shows that we can still obtain a unique solution for the parameters by solving

$$A g_T(\theta) = 0 \tag{4}$$

where $A$ denotes the $M \times L$ linear combinations of $g_T(\theta)$, whose optimal value, $A^*$, is obtained by minimizing the variance–covariance matrix of the parameter estimates, $\widehat{\theta}$. Specifically,

$$A^* = D_0 S_0^{-1} \tag{5}$$

where

$$D_0 = E\left[\frac{\partial h(X_t, \theta)}{\partial \theta}\right], \tag{6}$$

and

$$S_0 = \sum_{-\infty}^{\infty} E[h(X_t, \theta)h(X_{t-1}, \theta)'] \tag{7}$$

is the variance–covariance matrix of the mean of $h(X_t, \theta)$. For over-identified systems, a test based on the standardized $g_T(\theta)$ vector can be used to determine whether the model's unconditional moment restrictions hold. The test statistic is given by:

$$J_T = T g_T(\widehat{\theta})' S_0^{-1} g_T(\widehat{\theta}) \overset{asy}{\sim} \chi^2_{M-L} \tag{8}$$

The $J_T$ test is basically a goodness-of-fit test of a model that is estimated by GMM when the system is over-identified. A large value of the statistic implies that we can reject the null hypothesis that the unconditional moment restrictions implied by the MMD is supported by the data. The $J_T$ statistic is asymptotically $\chi^2$ with $M$-$L$ degrees of freedom.

We use the Hansen–Heaton–Ogaki GMM package to estimate the MMD parameters, based on sample estimates of $D_0$ and $S_0$.[2] Estimates of $S_0$ are computed using Newey–West autocorrelation and heteroskedastic-consistent positive definite estimators (Newey and West 1987).

## 4 Data

Our data consists of daily returns and trading volume of stocks that are components of the Dow Jones Industrial Average stock index. Data for returns and trading volume were obtained from the University of Chicago's Center for Research in Security Price (CRSP) database.

We examine two sample periods: 1973–1991 and 1993–1999. For the later period, data on number of trades is also available from the TAQ database supplied by the NYSE. Therefore, for this period, the model is tested using both total trading volume and number of trades.

Complete data for the two sample periods are available for 22 firms, all of which are large and actively traded, which is a necessary condition for testing the mixture of distributions model.

Daily returns were computed for each firm as the natural log of price relatives:

$$R_{it} = \ln\left(\frac{P_{it}}{P_{it-1}}\right) \tag{9}$$

where $P_{it}$ represents the closing price for stock $i$ on day $t$.

Trading volume is simply the total number of shares traded for each stock. Gallant, Rossi and Tauchen (1992) point out the need to de-trend the volume series since the mixture of distributions model cannot discriminate between increases in trading volume due to information arrivals and increases due to more traders in the market. Andersen (1996) used two de-trending methods. The first method de-trends each day's trading volume by dividing it by a normal value computed using nonparametric kernel regression. The other method estimates the normal volume using an equally weighted moving average of length 2 years centered on each day's trading volume. He shows that the results are robust to the two methods. This study uses the simpler moving average method and finds results which are similar to Andersen's. Figure 1 displays the time series of de-trended trading volume for the 22 stocks. The de-trended volume for all stocks appears to be stationary.[3]

## 5 Results

Table 1 reports the $\chi^2_3$ test statistic for the 22 stocks for 1973–1991, the same period as used by Andersen. The test statistic and $p$-values are shown in the second and third column, respectively. The last column shows the $p$-values of the five stocks

[2] Obtainable from the GAUSS archive at the American University website.

[3] As a check for stochastic trends, we apply the unit root test of Phillips and Perron (1988) to each series and find that the unit root null hypothesis is strongly rejected for all stocks.

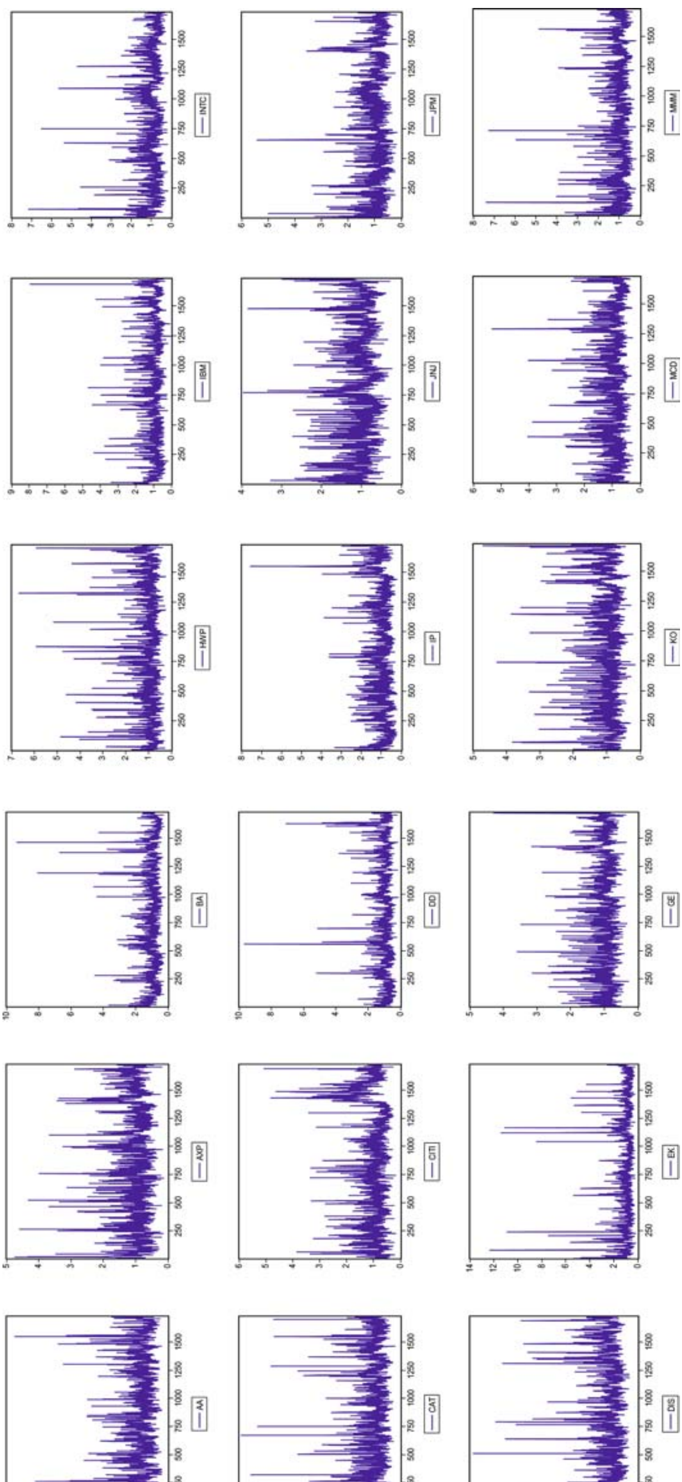


**Fig. 1** Detrended trading volume (1973–1991)

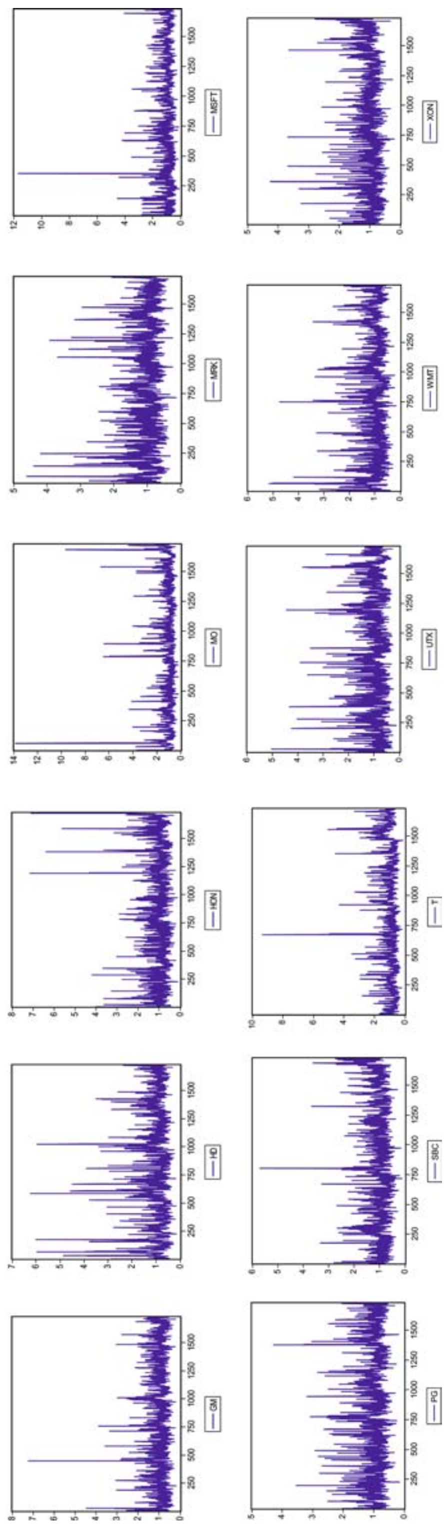


**Fig. 1** (Contd.)

**Table 1** Generalized method of moments (GMM) test of the modified mixture of distributions model (1973–1991)

| Stock | $\chi^2$ | $p$-value | Andersen (1996) |
|---|---|---|---|
| Alcoa | 6.92 | 0.074 | 0.074 |
| Amoco | 8.58 | **0.035** | **0.046** |
| American Express | 10.28 | **0.016** | |
| Boeing | 18.92 | **0.000** | |
| Caterpillar | 2.55 | 0.466 | |
| Coca-Cola | 7.39 | 0.061 | 0.066 |
| Du Pont | 7.6 | 0.055 | |
| Disney, Walt | 3.26 | 0.353 | |
| Eastman Kodak | 5.98 | 0.113 | 0.18 |
| Exxon-Mobil | 35.62 | **0.000** | |
| General Electric | 33.09 | **0.000** | |
| General Motors | 24.75 | **0.000** | |
| IBM | 4.45 | 0.220 | 0.23 |
| Int'nal Paper | 48.22 | **0.000** | |
| Morgan, JP & Co | 8.31 | **0.040** | |
| McDonalds | 8.52 | **0.040** | |
| Minnesota Mining & Mnfr | 42.57 | **0.000** | |
| Phillip Morris | 3.14 | 0.370 | |
| Merck | 2.67 | 0.411 | |
| Procter & Gamble | 43.3 | **0.000** | |
| AT & T | 12.02 | **0.007** | |
| United Technologies | 15.1 | **0.002** | |
| Mean | 16.06 | 0.10 | |
| Median | 8.55 | 0.04 | |

This table reports the GMM over-identification test for the Modified Mixture of Distributions model. The sample period is from January 1973 to December 1991. The test statistic is distributed $\chi^2$ with three degrees of freedom. Figures in bold indicate that the MMD is rejected at the 5% significance level

reported by Andersen. We note the following results. First, consistent with Andersen, the data supports the MMD for four of the five stocks (the exception being Amoco). If these stocks are representative of all liquid firms, one would be justified in concluding that the MMD has general empirical validity. However, this is not the case. Based on our full sample, we see that the MMD is rejected for a total of 13 stocks or 59% of the sample. Thus, at least for this sample period, we disagree with Andersen's claim that the MMD provides an acceptable characterization of the data.

Table 2 shows results for the more recent period (1993–1999). For this period, we include results for trading volume as well as number of trades. Compared to the earlier sample period, the data is more supportive of the MMD. Nonetheless 31% of the sample still rejects the MMD based on trading volume. In general, the test statistics are smaller for number of trades than for trading volume, suggesting that number of trades is a better proxy for information flow than volume per se. This result is consistent with Jones et al. (1994) who find that number of trades explains far more of daily stock returns volatility than trade size, and Ane and Geman (2000) who show that number of trades is the stochastic clock that recovers normality in the distribution of stock returns.

**Table 2** GMM test of the modified mixture of distributions model (1993–1999)

| Stock | Trading volume | | Number of trades | |
|---|---|---|---|---|
| | $\chi^2$ | $p$-value | $\chi^2$ | $p$-value |
| Alcoa | 11.62 | **0.009** | 17.69 | **0.000** |
| American Express | 11.52 | **0.009** | 11.52 | **0.009** |
| Amoco | 6.51 | 0.089 | 6.17 | 0.104 |
| Boeing | 4.19 | 0.242 | 3.72 | 0.294 |
| Caterpillar | 3.35 | 0.341 | 6.80 | 0.079 |
| Coca-cola | 8.1 | **0.044** | 1.78 | 0.619 |
| Du Pont | 6.67 | 0.083 | 4.20 | 0.241 |
| Disney, Walt | 2.25 | 0.523 | 0.80 | 0.851 |
| Eastman Kodak | 6.19 | 0.103 | 0.57 | 0.904 |
| Exxon-Mobil | 9.44 | **0.024** | 5.97 | 0.113 |
| General Electric | 3.68 | 0.298 | 2.88 | 0.411 |
| General Motors | 7.41 | 0.060 | 6.81 | 0.078 |
| IBM | 32.99 | **0.000** | 25.99 | **0.000** |
| Int'nal Paper | 12.9 | **0.005** | 18.45 | **0.000** |
| Morgan, JP & Co | 1.52 | 0.677 | 4.62 | 0.200 |
| McDonalds | 6.42 | 0.093 | 5.65 | 0.130 |
| Minnesota Mining & Mnfr | 2.29 | 0.514 | 5.91 | 0.116 |
| Phillip Morris | 6.19 | 0.103 | 7.02 | 0.071 |
| Merck | 28.97 | **0.000** | **11.93** | **0.008** |
| Procter & Gamble | 6.22 | 0.101 | 5.67 | 0.129 |
| AT & T | 3.88 | 0.275 | 11.93 | **0.008** |
| United Technologies | 2.3 | 0.512 | 2.48 | 0.478 |
| Mean | 8.39 | 0.19 | 7.66 | 0.22 |
| Median | 6.32 | 0.10 | 5.94 | 0.11 |

This table reports the GMM over-identification test for Modified Mixture of Distributions model. The sample period is from January 1993 to December 1999. The test statistic is distributed $\chi^2$ with three degrees of freedom. Figures in bold indicate that the MMD is rejected at the 5% significance level

## 6 Robustness checks

Are our results influenced by the volume de-trending method? Instead of experimenting with different arbitrary de-trending methods, we isolate stocks for which there is neither deterministic (linear and quadratic) trend nor stochastic (unit root) trends and hence do not require de-trending. Specifically, we fit a linear-quadratic trend model as in Gallant et al. (1992) to each stock. Stocks which do not have significant deterministic trend coefficients are then tested for the presence of stochastic trends using the Phillips–Perron unit root test. The results show that for the earlier sample period, all 22 stocks exhibit linear trends, with some also displaying quadratic trends. For the second sample period, we are able to isolate five stocks that are devoid of deterministic and stochastic trends. The five stocks are: Caterpillar, General Electric, General Motors, Merck, and United Technologies. Results for these five stocks are shown in Figure 2 and Table 3. For most stocks, the test statistics are quite similar to those based on de-trended volume, and the qualitative results are same as before. Thus, our main findings do not appear to be very sensitive to the volume de-trending method used.

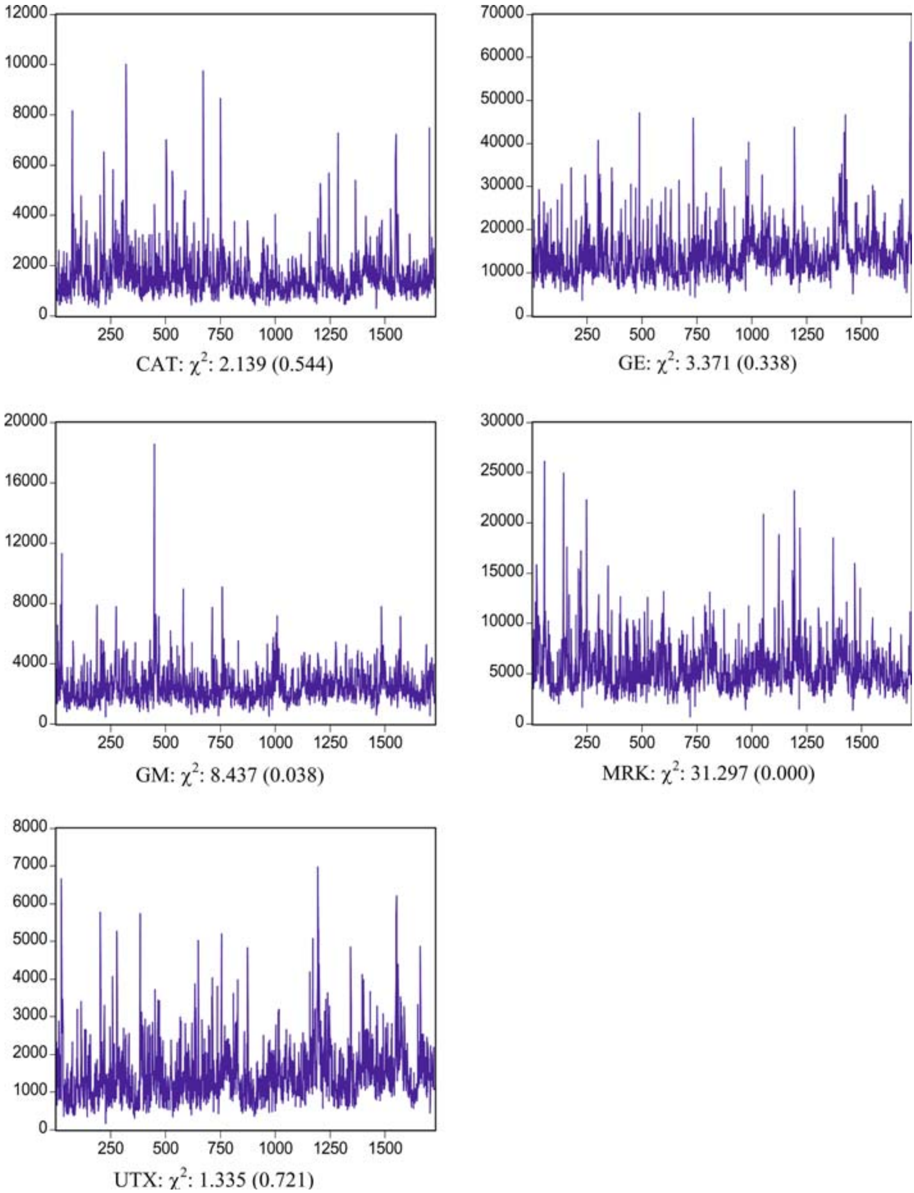


**Fig. 2** Raw trading volume (1993–1999)

## 7 Conclusion

We have examined the generality and robustness of the modified mixture model proposed by Andersen (1996). Using a larger sample of stocks and two sample periods, we test whether the modified mixture model provides an acceptable description of the volume–volatility relation. For the earlier sample period, the answer is no, as majority of stocks reject the implications of the model. Results for the more

**Table 3** GMM test of the modified mixture of distributions model based on raw trading volume

| Stock | De-trended volume | | Raw volume | |
|---|---|---|---|---|
| | $\chi^2$ | $p$-value | $\chi^2$ | $p$-value |
| Caterpillar | 3.35 | 0.341 | 2.14 | 0.544 |
| General Electric | 3.68 | 0.298 | 3.37 | 0.338 |
| General Motors | 7.41 | 0.060 | 8.44 | 0.038 |
| Merck | 28.97 | 0.000 | 31.30 | 0.000 |
| United Technologies | 2.30 | 0.512 | 1.34 | 0.721 |
| Mean | 9.14 | 0.24 | 9.32 | 0.33 |
| Median | 3.68 | 0.30 | 3.37 | 0.34 |

This table compares GMM over-identification test results for five stocks using de-trended and raw trading volume. The sample period is from January 1993 to December 1999. The test statistic is distributed $\chi^2$ with three degrees of freedom

recent period are more encouraging, although there is still a rejection rate of about 30%. Therefore, it is difficult to make a compelling case for the model as a robust description of the volume–volatility relation at this point. Nonetheless, the MMD is a substantive improvement over the classical mixture of distributions model and it deserves further empirical validation. We leave this validation as a task for future research.

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